Exercices et problèmes

1° a) $\int_{-1}^{2} 3t^2\,dt = \left[t^3\right]_{-1}^{2} = 9$.

b) $\int_{0}^{\pi} 8\cos x\,dx = \left[8\sin x\right]_{0}^{\pi} = 0$.

c) $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} 4\sin x\,dx = \left[-4\cos x\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}} = 0$.

d) $\int_{1}^{3} \frac{4dt}{t^2} = \left[-\frac{4}{t}\right]_{1}^{3} = \frac{8}{3}$.

2° a) $\int_{-1}^{0} (4t^3 - 6t + 5)\,dt = \left[t^4 - 3t^2 + 5t\right]_{-1}^{0} = 7$.

b) $\int_{2}^{-8} 0dt = 0$.

c) $\int_{1}^{-4} 4dt = \left[4t\right]_{1}^{-4} = -20$.

d) $\int_{0}^{\frac{\pi}{4}} (1 + \tan^2 z)\,dz = \left[\tan z\right]_{0}^{\frac{\pi}{4}} = 1$.

3° a) $\int_{0}^{\frac{\pi}{10}} 8\cos 5x\,dx = \left[\frac{8}{5}\sin 5x\right]_{0}^{\frac{\pi}{10}} = \frac{8}{5}$.

b) $\int_{0}^{4\pi} 2\sin\frac{x}{8}\,dx = \left[-16\cos\frac{x}{8}\right]_{0}^{4\pi} = 16$.

c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (5\cos 2x - 8\sin 3x)\,dx = \left[\frac{5}{2}\sin 2x + \frac{8}{3}\cos 3x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -\frac{8}{3}$.

d) $\int_{0}^{\frac{\pi}{4}} \tan^2 t\,dt = \int_{0}^{\frac{\pi}{4}} [1 + \tan^2 t - 1]\,dt = [\tan t - t]_{0}^{\frac{\pi}{4}} = 1 - \frac{\pi}{4}$.

e) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 t\,dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2}\,dt = \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{1}{2}\left[\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right] = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$.

4° a) $\int_{0}^{1} \frac{2x\,dx}{\sqrt{4 - x^2}} = \left[-2\sqrt{4 - x^2}\right]_{0}^{1} = -2\sqrt{3} + 4$.

b) $\int_{2}^{3} \frac{x\,dx}{(x^2 - 1)^2} = \left[-\frac{1}{2(x^2 - 1)}\right]_{2}^{3} = -\frac{1}{2}\left(\frac{1}{8} - \frac{1}{3}\right) = \frac{5}{48}$.

c) $\int_{0}^{1} x\sqrt{1 + x^2}\,dx = \int_{0}^{1} x(1 + x^2)^{\frac{1}{2}}\,dx = \left[\frac{1}{3}(1 + x^2)^{\frac{3}{2}}\right]_{0}^{1} = \frac{2\sqrt{2} - 1}{3}$.

d) $\int_{0}^{4} \frac{dx}{\sqrt{2x + 1}} = \left[\sqrt{2x + 1}\right]_{0}^{4} = 3 - 1 = 2$.

5° **a)** $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\cos x}{(1+\sin x)^2}\,dx = \left[\frac{-1}{1+\sin x}\right]_0^{\frac{\pi}{2}} = -\frac{1}{2} + 1 = \frac{1}{2}$.

b) $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2x \cos x\,dx = \int_0^{\frac{\pi}{2}} 2\sin x \cos^2 x\,dx = \left[-\frac{2}{3}\cos^3 x\right]_0^{4} = \frac{2}{3}$.

c) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\tan x}{\cos^4 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^5 x}\,dx = \left[\frac{1}{4\cos^4 x}\right]_0^{\frac{\pi}{4}} = \frac{3}{4}$.

d) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos^3 x}\,dx = \left[\frac{1}{2\cos^2 x}\right]_0^{\frac{\pi}{3}} = \frac{3}{2}$.

6° **a)** $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2\sin x}{\cos^2 x}\,dx = \left[\frac{2}{\cos x}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 2\left(2 - \frac{2}{\sqrt{3}}\right) = 4 - \frac{4}{\sqrt{3}}$.

b) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos z}{\sin^4 z}\,dz = \left[-\frac{1}{3\sin^3 z}\right] = \frac{-1}{3}\,(1 - 2\sqrt{2}) = \frac{2\sqrt{2}-1}{3}$.

c) $\displaystyle\int_0^{\frac{\pi}{4}} (1+\tan^2 x)\tan^5 x\,dx = \left[\frac{1}{6}\tan^6 x\right]_0^{\frac{\pi}{4}} = \frac{1}{6}$.

d) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{dt}{\cos^2 t} = \left[\tan t\right]_0^{\frac{\pi}{3}} = \sqrt{3}$.

7° **a)** $\displaystyle\int_0^{\frac{\pi}{12}} (1+\tan^2 3x)\tan^{1999} 3x\,dx = \int_0^{\frac{\pi}{12}} \frac{1}{3}\,(1+\tan^2 3x)\,.\,\tan^{1999} 3x\,.\,3\,dx$

$\displaystyle = \left[\frac{1}{3.2000}\tan^{2000} 3x\right]_0^{\frac{\pi}{12}} = \frac{1}{6000}$.

b) $\displaystyle\int_0^{1} x(1+x^2)^{1999}\,dx = \int_0^{1} \frac{1}{2}\,(1+x^2)^{1999}\,2x\,dx = \left[\frac{1}{2}\,\frac{1}{2000}\,(1+x^2)^{2000}\right]_0^{1}$

$\displaystyle = \frac{1}{4000}\,(2^{2000}-1)$.

c) $\displaystyle\int_0^{\frac{\pi}{2}} \cos x \sin^{1999} x\,dx = \left[\frac{1}{2000}\sin^{2000} x\right]_0^{\frac{\pi}{2}} = \frac{1}{2000}$.

8° **a)** $\displaystyle\int_0^{\frac{\pi}{12}} \sin 2x \cos 2x\,dx = \int_0^{\frac{\pi}{12}} \frac{1}{2}\sin 4x\,dx = \left[-\frac{1}{8}\cos 4x\right]_0^{\frac{\pi}{12}} = \frac{1}{16}$.

b) $\displaystyle\int_0^{1} \frac{8t}{\sqrt{1+4t^2}}\,dt = \int_0^{1} 2\,\frac{8t}{2\sqrt{1+4t^2}}\,dt = \left[2\sqrt{1+4t^2}\right]_0^{1} = 2(\sqrt{5}-1)$.

c) $\displaystyle\int_0^{2} \frac{x+1}{\sqrt{x^2+2x+8}}\,dx = \int_0^{2} \frac{2x+2}{2\sqrt{x^2+2x+8}}\,dx = \left[\sqrt{x^2+2x+8}\right]_0^{2} = 4 - 2\sqrt{2}$.

9° **a)** $$\int_0^{\frac{\pi}{40}} (\cos^2 x + \sin^2 x)\, dx = \int_0^{\frac{\pi}{40}} dx = \Big[x\Big]_0^{\frac{\pi}{40}} = \frac{\pi}{40}.$$

b) $$\int_0^{\frac{\pi}{8}} (\cos^2 x - \sin^2 x)\, dx = \int_0^{\frac{\pi}{8}} \cos 2x\, dx = \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{8}} = \frac{\sqrt{2}}{4}.$$

c) $$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\cot 3x}{\sin 3x}\, dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\cos 3x}{\sin^2 3x}\, dx = \left[\frac{-1}{3\sin 3x}\right]_{\frac{\pi}{12}}^{\frac{\pi}{6}} = -\frac{1}{3}\left[1-\sqrt{2}\right] = \frac{\sqrt{2}-1}{3}.$$

d) $$\int_0^{\frac{\pi}{2}} \frac{\sin x\, dx}{\sqrt{3+\cos x}} = \int_0^{\frac{\pi}{2}} -2\,\frac{-\sin x}{2\sqrt{3+\cos x}}\, dx = \left[-2\sqrt{3+\cos x}\right]_0^{\frac{\pi}{2}} = -2\sqrt{3}+4.$$

[illegible] 1° $\int_2^3 8(4x-1)(2x^2-x-5)^4\,dx = \left[\frac{8}{5}(2x^2-x-5)^5\right]_2^3 = \frac{8}{5}(10^5-1)$.

2° $\int_2^3 \frac{2t+1}{\sqrt{t^2+t-1}}\,dt = \left[2\sqrt{t^2+t-1}\right]_2^3 = 2\sqrt{11}-2\sqrt{5}$.

3° $\int_0^{\frac{\pi}{2}} \sin 2x \cos^{25} x\,dx = \int_0^{\frac{\pi}{2}} 2\sin x \cos^{26} x\,dx = \left[\frac{-2}{27}\cos^{27} x\right]_0^{\frac{\pi}{2}} = \frac{2}{27}$.

4° $\int_1^2 \frac{10x+1}{(5x^2+x+3)^2}\,dx = \left[-\frac{1}{5x^2+x+3}\right]_1^2 = \frac{-1}{25}+\frac{1}{9} = \frac{16}{225}$.

5° $\int_0^{\frac{\pi}{8}} (1+\tan^2 2x)\tan^{35} 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{8}} 2(1+\tan^2 2x)\tan^{35} 2x\,dx = \frac{1}{2}\left[\frac{1}{36}\tan^{36} 2x\right]_0^{\frac{\pi}{8}}$

$= \frac{1}{72}$.

6° $\int_0^{\frac{\pi}{4}} \frac{8\sin 2x}{\cos^4 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{8 \,.\, 2\sin x\cos x}{\cos^4 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{16\sin x}{\cos^4 x}\,dx = \left[+\frac{16}{3}\frac{1}{\cos^3 x}\right]_0^{\frac{\pi}{4}}$

$= \frac{16}{3}\left(2\sqrt{2}-1\right)$

7° $\int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{(1+\sin x)^5} = \left[-\frac{1}{4(1+\sin x)^4}\right]_0^{\frac{\pi}{2}} = \frac{-1}{4 \,.\, 2^4}+\frac{1}{4} = \frac{1}{4}-\frac{1}{2^6}$.

8° $\int_{-1}^1 t\sqrt{1+t^2}\,dt = 0$.

9° $\int_0^{\frac{\pi}{4}} \frac{3\sin x}{(1+\cos x)^4}\,dx = \left[\frac{3}{3}\frac{1}{(1+\cos x)^3}\right]_0^{\frac{\pi}{4}} = \frac{2\sqrt{2}}{(1+\sqrt{2})^3}-\frac{1}{8}$.

[illegible] 1° $\int_{-1}^1 |x|\,dx = \int_{-1}^0 -x\,dx + \int_0^1 x\,dx = \left[-\frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^2}{2}\right]_{-1}^0 = 1$.

2° $\int_{-2}^1 |x^2-1|\,dx = \int_{-2}^{-1} (x^2-1)\,dx + \int_{-1}^1 (1-x^2)\,dx = \left[\frac{x^3}{3}-x\right]_{-2}^1 + \left[x-\frac{x^3}{3}\right]_0^1$

$= \left(\frac{-1}{3}+1\right) - \left(\frac{-8}{3}+2\right) + \left(1-\frac{1}{3}\right) = \frac{6}{3} = 2$.

3° $\int_0^{\pi} \sin x \left|\cos^3 x\right| dx = \int_0^{\frac{\pi}{2}} \sin x \cos^3 x \, dx + \int_{\frac{\pi}{2}}^{\pi} - \sin x \cos^3 x \, dx$

$$= \left[\frac{-1}{4}\cos^4 x\right]_0^{\frac{\pi}{2}} + \left[\frac{1}{4}\cos^4 x\right]_{\frac{\pi}{2}}^{\pi} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

4° $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left|x \cos x\right| dx = 2\int_0^{\frac{\pi}{4}} x \cos x \, dx$ car la fonction $x \mapsto |x \cos x|$ est paire dans l'intervall $\left[\frac{-\pi}{4} ; \frac{\pi}{4}\right]$.

$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left|x \cos x\right| dx = 2\int_0^{\frac{\pi}{4}} x \cos x \, dx = 2 \int_0^{\frac{\pi}{4}} u'(x)\, v(x)\, dx$; avec :

$u'(x) = \cos x$ et $v(x) = x$, alors $u(x) = \sin x$ et $v'(x) = 1$.

$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left|x \cos x\right| dx = 2\left[\left[x \sin x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin x \, dx\right] = 2\left[\left[x \sin x + \cos x\right]_0^{\frac{\pi}{4}}\right]$

$$= \frac{\pi\sqrt{2}}{4} + \sqrt{2} - 2 .$$

5° $\int_{-1}^{2} \frac{|t|}{\sqrt{1+t^2}} dt = \int_{-1}^{1} \frac{|t|}{\sqrt{1+t^2}} dt + \int_{1}^{2} \frac{t}{\sqrt{1+t^2}} dt$

$= 2\int_0^1 \frac{t}{\sqrt{1+t^2}} dt + \int_1^2 \frac{t}{\sqrt{1+t^2}} dt$ car la fonction $t \mapsto \frac{|t|}{\sqrt{1+t^2}}$ est paire sur l'intervalle $[-1 ; 1]$.

$$\int_{-1}^{2} \frac{|t|}{\sqrt{1+t^2}} dt = \left[2\sqrt{1+t^2}\right]_0^1 + \left[\sqrt{1+t^2}\right]_1^2 = \sqrt{5} + \sqrt{2} - 2 .$$

6° $\int_0^{\pi} \left|\pi - 3z\right| \sin z \, dz = \int_0^{\frac{\pi}{3}} (\pi - 3z) \sin z \, dz + \int_{\frac{\pi}{3}}^{\pi} (3z - \pi) \sin z \, dz$ or $\int_{\alpha}^{\beta} (\pi - 3z) \sin z$

$$= \int_{\alpha}^{\beta} u'(z)\, v(z)\, dx \text{ ; avec :}$$

$u'(z) = \sin z$ et $v(z) = \pi - 3z$, alors $u(z) = -\cos z$ et $v'(z) = -3$.

$$\int_{\alpha}^{\beta} [\pi - 3z] \sin z \, dz = \left[-(\pi - 3z)\cos z\right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} 3 \sin z \, dz = \left[-(\pi - 3z) + 3\cos z\right]_{\alpha}^{\beta}.$$

$$\int_0^{\pi} \left|\pi - 3z\right| \sin z \, dz = \left[-(\pi - 3z)\cos z + 3\cos z\right]_0^{\frac{\pi}{3}} + \left[(\pi - 3z)\cos z - 3\cos z\right]_{\frac{\pi}{3}}^{\pi}$$
$$= \pi - \frac{3}{2} .$$

7° $\displaystyle\int_{-6}^{2}\left|x^2+4x-5\right|dx=\int_{-6}^{2}\left|(x-1)(x+5)\right|dx$

$$=\int_{-6}^{-5}(x^2+4x-5)\,dx+\int_{-5}^{1}-(x^2+4x-5)\,dx+\int_{1}^{2}(x^2+4x-5)\,dx$$

$$=\left[\frac{x^3}{3}+2x^2-5x\right]_{-6}^{-5}+\left[-\left(\frac{x^3}{3}+2x^2-5x\right)\right]_{-5}^{1}+\left[\frac{x^3}{3}+2x^2-5x\right]_{1}^{2}$$

$$=\frac{128}{3}\,.$$

8° $\displaystyle\int_{-1}^{2}\frac{|x^3|}{(1+x^2)^3}\,dx=\int_{-1}^{1}\frac{|x^3|}{(1+x^2)^3}\,dx+\int_{1}^{2}\frac{x^3}{(1+x^2)^3}\,dx$

$$=2\int_{0}^{1}\frac{x^3}{(1+x^2)^3}\,dx+\int_{1}^{2}\frac{x^3}{(1+x^2)^3}\,dx$$

car la fonction $x\mapsto\dfrac{|x^3|}{(1+x^2)^3}$ est paire sur l'intervalle $[-1\ ;\ 1]$.

$$\int_{-1}^{2}\frac{|x^3|}{(1+x^2)^3}\,dx=2\int_{0}^{1}\frac{x(x^2+1)-x}{(1+x^2)^3}\,dx+\int_{1}^{2}\frac{x(x^2+1)-x}{(1+x^2)^3}\,dx$$

$$=2\int_{0}^{1}\left[\frac{x}{(1+x^2)^2}-\frac{x}{(1+x^2)^3}\right]dx+\int_{1}^{2}\left[\frac{x}{(1+x^2)^2}-\frac{x}{(1+x^2)^3}\right]dx$$

$$=2\left[\frac{-1}{2(1+x^2)}+\frac{1}{4(1+x^2)^2}\right]_{0}^{1}+\left[\frac{-1}{2(1+x^2)}+\frac{1}{4(1+x^2)^2}\right]_{1}^{2}=\frac{89}{400}\,.$$

9° $\displaystyle\int_{\frac{-\pi}{4}}^{\frac{\pi}{2}}|x|\,|\sin x|\,dx=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}|x|\,|\sin x|\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\sin x\,dx=2\int_{0}^{\frac{\pi}{4}}x\sin x\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\sin x\,dx$

voir numéro 4) **1°**

$$=2\Big[-x\cos x+\sin x\Big]_{0}^{\frac{\pi}{4}}+\Big[-x\cos x+\sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=1+\frac{\sqrt{2}}{2}-\frac{\pi\sqrt{2}}{8}\,.$$

11 1°