9 $u = 2 - 2i\sqrt{3}$

1° $u = 4\left(\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right) = 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = \left[4\,;-\frac{\pi}{3}\right] = 4e^{-i\frac{\pi}{3}}$

$z^4 = u$ équivaut à $z^4 = 4e^{-i\frac{\pi}{3}}$

$z_0 = \sqrt{2}\,e^{-i\frac{\pi}{12}}$; $z_1 = \sqrt{2}\,e^{i\left(-\frac{\pi}{12} + \frac{\pi}{2}\right)}$; $z_2 = \sqrt{2}\,e^{i\left(\frac{-\pi}{12} + \pi\right)}$;

$z_3 = \sqrt{2}\,e^{i\left(\frac{-\pi}{12} + \frac{3\pi}{2}\right)}$.

2° On pose $X = a + ib$, on obtient :

$(a + ib)^2 = 2 - 2i\sqrt{3}$ et $|a + ib|^2 = |2 - 2i\sqrt{3}|$

$\begin{cases} a^2 - b^2 + 2iab = 2 - 2i\sqrt{3} \\ a^2 + b^2 = \sqrt{4 + 12} \end{cases}$; $\begin{cases} a^2 - b^2 = 2 \\ a^2 + b^2 = 4 \\ ab = -\sqrt{3} \end{cases}$

$a = \sqrt{3}$, $b = -1$ ou $a = -\sqrt{3}$, $b = 1$.

$X_1 = \sqrt{3} - i$ ou $X_2 = -\sqrt{3} + i = -X_1$

$z = \alpha + i\beta$ avec $z^2 = X = \sqrt{3} - i$ donne :

$\begin{cases} \alpha^2 - \beta^2 + 2i\alpha\beta = \sqrt{3} - i \\ \alpha^2 + \beta^2 = \sqrt{3 + 1} = 2 \end{cases}$ $\begin{cases} \alpha^2 - \beta^2 = \sqrt{3} \\ \alpha^2 + \beta^2 = 2 \\ \alpha\beta = -\frac{1}{2} \end{cases}$

$\alpha^2 = \frac{\sqrt{3} + 2}{2}$, $\beta^2 = \frac{2 - \sqrt{3}}{2}$, $\alpha\beta = -\frac{1}{2}$

$\left(\alpha = \sqrt{\frac{2 + \sqrt{3}}{2}},\ \beta = -\sqrt{\frac{2 - \sqrt{3}}{2}}\right)$ ou $\left(\alpha = -\sqrt{\frac{-2 + \sqrt{3}}{2}},\ \beta = \sqrt{\frac{2 - \sqrt{3}}{2}}\right)$

On en déduit :

$z_0 = \sqrt{\frac{2 + \sqrt{3}}{2}} - i\sqrt{\frac{2 - \sqrt{3}}{2}}$; $z_1 = -\sqrt{\frac{2 + \sqrt{3}}{2}} + i\sqrt{\frac{2 - \sqrt{3}}{2}}$

Or $X_2 = -X_1 = i^2 X_1$ d'où

$z_2 = z_0 \,.\, i = \sqrt{\frac{2 - \sqrt{3}}{2}} + i\sqrt{\frac{2 + \sqrt{3}}{2}}$, $z_3 = -\sqrt{\frac{2 - \sqrt{3}}{2}} - i\sqrt{\frac{2 + \sqrt{3}}{2}}$.

3° $\cos\frac{\pi}{12} > 0$ et $\sin\frac{\pi}{12} > 0$

$\sqrt{2}\,e^{-i\frac{\pi}{12}} = \sqrt{2}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$ donne : $\sqrt{2}\,e^{-i\frac{\pi}{12}} = z_0$, et

$\cos\frac{\pi}{12} = \frac{1}{\sqrt{2}}\sqrt{\frac{2 + \sqrt{3}}{2}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$, $\sin\frac{\pi}{12} = \frac{1}{\sqrt{2}}\sqrt{\frac{2 - \sqrt{3}}{2}} = \frac{\sqrt{2 - \sqrt{3}}}{2}$.